Parallel Systems Course: Chapter IX

Sorting Algorithms

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Overview

1. Parallel sort – distributed memory
2. Parallel sort – shared memory
3. Sorting Networks
   A. Odd-even
   B. Bitonic
1. Parallel sort – distributed memory
2. Parallel sort – shared memory
3. Sorting Networks
   A. Odd-even
   B. Bitonic
Mission

Sort array *asap* by exploiting parallel system with distributed memory

*Idea*: based on quicksort

leads to most-optimal parallel algorithm?
Quicksort

- Quicksort is one of the most common sorting algorithms for sequential computers because of its simplicity, low overhead, and optimal average complexity.

- Quicksort selects one of the entries in the sequence to be the pivot and divides the sequence into two - one with all elements less than the pivot and other greater.

- The process is recursively applied to each of the sublists.
The sequential quicksort algorithm.
Quicksort

Example of the quicksort algorithm sorting a sequence of size $n = 8$. 
Quicksort

- The performance of quicksort depends critically on the quality of the pivot.
- In the best case, the pivot divides the list in such a way that the larger of the two lists does not have more than \(\alpha n\) elements (for some constant \(\alpha\)).
- In this case, the complexity of quicksort is \(O(n \log n)\).
v1. Parallel Quicksort

- Lets start with recursive decomposition - the list is partitioned serially and each of the subproblems is handled by a different processor.

- The time for this algorithm is lower-bounded by $\Omega(n)$!
  - Since the partitioning is done on single processor

- Can we parallelize the partitioning step - in particular, if we can use $n$ processors to partition a list of length $n$ around a pivot in $O(1)$ time, we have a winner.
  - Then we obtain a runtime of $O(\log n)$!!

- This is difficult to do on real machines, though.
Parallel Quicksort

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Execution Profile

Parallel Sorting

- Communication
- Blocking (Idling)
- Computing
Can we resolve the load imbalances?

- Make sure that each processor has the same number of elements locally.
- Merge results
- Merge sort!
  - Actually better than quicksort
  - Disadvantage: not in place (need copy of matrix)
    - Use quicksort for local sort
v2. based on merge sort

Similar communication overhead, but without load imbalances!
v3. Can we overcome the limited parallelism in the beginning?

- A simple message passing formulation is based on the recursive halving of the machine.
- Assume that each processor in the lower half of a \( p \) processor ensemble is paired with a corresponding processor in the upper half.
- A designated processor selects and broadcasts the pivot.
- Each processor splits its local list into two lists, one less \( (L_i) \), and other greater \( (U_i) \) than the pivot.
- A processor in the low half of the machine sends its list \( U_i \) to the paired processor in the other half. The paired processor sends its list \( L_i \).
After this step:
- all elements < pivot in the low half of the machine
- all elements > pivot in the high half.

The above process is recursed until each processor has its own local list, which is sorted locally.

The time for a single reorganization is $\Theta(\log p)$ for broadcasting the pivot element, $\Theta(n/p)$ for splitting the locally assigned portion of the array, $\Theta(n/p)$ for exchange and local reorganization.

Note that this time is identical to that of the corresponding shared address space formulation.

However, it is important to remember that the reorganization of elements is a bandwidth sensitive operation.
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Parallelizing Quicksort: Shared Address Space Formulation

- A list of size $n$ equally divided across $p$ processors.
- A pivot is selected by one of the processors and made known to all processors.
- Each processor partitions its list into two, say $L_i$ and $U_i$, based on the selected pivot.
- All of the $L_i$ lists are merged and all of the $U_i$ lists are merged separately.
- The set of processors is partitioned into two (in proportion of the size of lists $L$ and $U$). The process is recursively applied to each of the lists.
Parallel Sorting

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**Remaining problem**: global reorganization (merging) of local lists to form $L$ and $U$.

- The problem is one of determining the right location for each element in the merged list.
- Each processor computes the number of elements locally less than and greater than pivot.
- It computes two *sum-scans* (also called *prefix sum*) to determine the starting location for its elements in the merged $L$ and $U$ lists.
- Once it knows the starting locations, it can write its elements safely.
Scan operation

*Parallel prefix sum:* every node got sum of previous nodes + itself
Efficient global rearrangement of the array.
The parallel time depends on the split and merge time, and the quality of the pivot.

The latter is an issue independent of parallelism, so we focus on the first aspect, assuming ideal pivot selection.

One iteration has four steps: (i) determine and broadcast the pivot; (ii) locally rearrange the array assigned to each process; (iii) determine the locations in the globally rearranged array that the local elements will go to; and (iv) perform the global rearrangement.

The first step takes time $\Theta(\log p)$, the second, $\Theta(n/p)$, the third, $\Theta(\log p)$, and the fourth, $\Theta(n/p)$.

The overall complexity of splitting an $n$-element array is $\Theta(n/p) + \Theta(\log p)$. 

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Parallelizing Quicksort: Shared Address Space Formulation

- The process recurses until there are $p$ lists, at which point, the lists are sorted locally.
- Therefore, the total parallel time is:

$$T_P = \Theta \left( \frac{n}{p} \log \frac{n}{p} \right) + \Theta \left( \frac{n}{p} \log p \right) + \Theta(\log^2 p).$$  \hspace{1cm} (4)
Alternative: PRAM Formulation

- We assume a **CRCW** (concurrent read, concurrent write) PRAM with concurrent writes resulting in an *arbitrary write succeeding* (!!).

- The formulation works by creating pools of processors. Every processor is assigned to the same pool initially and has one element.

- Each processor attempts to write its element to a common location (for the pool).

- Each processor tries to read back the location. If the value read back is greater than the processor's value, it assigns itself to the `left' pool, else, it assigns itself to the `right' pool.

- Each pool performs this operation recursively.

- Note that the algorithm generates a tree of pivots. The depth of the tree is the expected parallel runtime. The average value is $O(\log n)$. 
A binary tree generated by the execution of the quicksort algorithm. Each level of the tree represents a different array-partitioning iteration. If pivot selection is optimal, then the height of the tree is $\Theta(\log n)$, which is also the number of iterations.
The execution of the PRAM algorithm on the array shown in (a).
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Mission

- Digital circuit that transforms an unsorted list (input) into a sorted list (output)
- Idea: parallel processing! By putting components in parallel (width)!!
- So: runtime is determined by depth
- Goal: minimal depth
Sorting Networks

 Networks of comparators designed specifically for sorting (time $< \Theta(n \log n)$).

 Specific-designed parallel system.

 A comparator is a device with two inputs $x$ and $y$ and two outputs $x'$ and $y'$. For an increasing comparator, $x' = \min\{x, y\}$ and $y' = \max\{x, y\}$; and vice-versa for a decreasing comparator.

 We denote an increasing comparator by $\oplus$ and a decreasing comparator by $\ominus$.

 The speed of the network is proportional to its depth.
Basic component: Comparators

A schematic representation of comparators: (a) an increasing comparator, and (b) a decreasing comparator.
A typical sorting network. Every sorting network is made up of a series of columns, and each column contains a number of comparators connected in parallel.
Best algorithm to hardwire?

Can we sort $n$ elements in time $O(\log n)$?

= quicksort performance

Quicksort not possible: communication paths are not fixed

Best: using $O(n \cdot \log n)$ comparators, but with a quite large constant (many thousands)

Not practical

**Bitonic sort** and **odd-even sort**: sort $n$ elements in time $O(\log^2 n)$
Sorting: Overview

- One of the most commonly used and well-studied kernels.
- Sorting can be comparison-based or noncomparison-based.
  - Noncomparison: determine rank (index) in list
- We focus here on comparison-based sorting algorithms.
- The fundamental operation of comparison-based sorting is compare-exchange.
- The lower bound on any comparison-based sort of \( n \) numbers is \( \Theta(n \log n) \).
A parallel compare-exchange operation. Processes $P_i$ and $P_j$ send their elements to each other. Process $P_i$ keeps $\min\{a_i, a_j\}$, and $P_j$ keeps $\max\{a_i, a_j\}$. 
What is the parallel counterpart to a sequential comparator?

- If each processor has one element, the compare exchange operation can be done in $t_s + t_w$ time (startup latency and per-word time).

- If we have more than one element per processor, we call this operation a **compare split**. Assume each of two processors have $n/p$ elements.
  
  - After the compare-split operation, the smaller $n/p$ elements are at processor $P_i$ and the larger $n/p$ elements at $P_j$, where $i < j$.
  
  - The time for a compare-split operation is $(t_s + t_w n/p)$, assuming that the two partial lists were initially sorted.
A compare-split operation. Each process sends its block of size $n/p$ to the other process. Each process merges the received block with its own block and retains only the appropriate half of the merged block. In this example, process $P_i$ retains the smaller elements and process $P_j$ retains the larger elements.

There are alternatives! With more communication, however…
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Bubble Sort and its Variants

The sequential bubble sort algorithm compares and exchanges adjacent elements in the sequence to be sorted:

1. procedure BUBBLE_SORT(n)
2. begin
3. for i := n - 1 downto 1 do
4. for j := 1 to i do
5. compare-exchange(a_j, a_{j+1});
6. end BUBBLE_SORT

Sequential bubble sort algorithm.
Bubble Sort and its Variants

- The complexity of bubble sort is $\Theta(n^2)$.
- Bubble sort is difficult to parallelize since the algorithm has no concurrency.
- A simple variant, though, uncovers the concurrency.
  - Complexity is lower than quicksort, but parallelization is more efficient.
Odd-Even Transposition

1. procedure ODD-EVEN(n)
2. begin
3. for i := 1 to n do
4. begin
5. if i is odd then
6. for j := 0 to n/2 − 1 do
7. compare-exchange(a_{2j+1}, a_{2j+2});
8. if i is even then
9. for j := 1 to n/2 − 1 do
10. compare-exchange(a_{2j}, a_{2j+1});
11. end for
12. end ODD-EVEN

Sequential odd-even transposition sort algorithm.
Sorting $n = 8$ elements, using the odd-even transposition sort algorithm. During each phase, $n = 8$ elements are compared.
Odd-Even Transposition

- After $n$ phases of odd-even exchanges, the sequence is sorted.
- Each phase of the algorithm (either odd or even) requires $\Theta(n)$ comparisons.
- Serial complexity is $\Theta(n^2)$.
- Parallel version can be implemented by 1 network which is used iteratively!

Conclusion: very simple, but not the fastest
Consider the **one item per processor** case.

There are $n$ iterations, in each iteration, each processor does one compare-exchange.

The parallel run time of this formulation is $\Theta(n)$.

This is cost optimal with respect to the base serial algorithm but not to the optimal one ($\Theta(n \log n)$).
procedure ODD-EVEN_PAR(n)
begin
  id := process's label
  for i := 1 to n do
    begin
      if i is odd then
        if id is odd then
          compare-exchange_min(id + 1);
        else
          compare-exchange_max(id - 1);
        if i is even then
          if id is even then
            compare-exchange_min(id + 1);
          else
            compare-exchange_max(id - 1);
    end for
end ODD-EVEN_PAR

Parallel formulation of odd-even transposition.
Parallel Odd-Even Transposition

- Consider a block of \( n/p \) elements per processor.
- The first step is a local sort.
- In each subsequent step of \( p \) steps, the compare exchange operation is replaced by the compare split operation (\( n/p \) comparisons).
- The parallel run time of the formulation is

\[
T_P = \Theta \left( \frac{n}{p} \log \frac{n}{p} \right) + \Theta(n) + \Theta(n).
\]
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A bitonic sorting network sorts $n$ elements in $\Theta(\log^2 n)$ time.

A **bitonic sequence** has two tones - increasing and decreasing, or vice versa.

$\langle 1,2,4,7,6,0 \rangle$ is a bitonic sequence, because it first increases and then decreases.

*Not important here:* Any cyclic rotation of a two-tone sequence is also considered bitonic. $\langle 8,9,2,1,0,4 \rangle$ is another bitonic sequence, because it is a cyclic shift of $\langle 0,4,8,9,2,1 \rangle$.

The kernel of the network is the rearrangement of a bitonic sequence into a sorted sequence.
Let $s = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be a bitonic sequence such that $a_0 \leq a_1 \leq \cdots \leq a_{n/2-1}$ and $a_{n/2} \geq a_{n/2+1} \geq \cdots \geq a_{n-1}$.

Consider the following subsequences of $s$:

$$s_1 = \langle \min\{a_0, a_{n/2}\}, \min\{a_1, a_{n/2+1}\}, \ldots, \min\{a_{n/2-1}, a_{n-1}\} \rangle$$

$$s_2 = \langle \max\{a_0, a_{n/2}\}, \max\{a_1, a_{n/2+1}\}, \ldots, \max\{a_{n/2-1}, a_{n-1}\} \rangle$$

(1)

$s_1$ and $s_2$ are both bitonic and each element of $s_1$ is less than every element in $s_2$.

We can apply the procedure recursively on $s_1$ and $s_2$ to get the sorted sequence.
Bitonic sort’s basic merge component

**Basic operation:** change a bitonic array into a sorted array. For 16 elements this can be done in 4 steps.

<table>
<thead>
<tr>
<th>Original sequence</th>
<th>3 5 8 9 10 12 14 20 95 90 60 40 35 23 18 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Split</td>
<td>3 5 8 9 10 12 14 0 95 90 60 40 35 23 18 20</td>
</tr>
<tr>
<td>2nd Split</td>
<td>3 5 8 0 10 12 14 9 35 23 18 20 95 90 60 40</td>
</tr>
<tr>
<td>3rd Split</td>
<td>3 0 8 5 10 9 14 12 18 20 35 23 60 40 95 90</td>
</tr>
<tr>
<td>4th Split</td>
<td>0 3 5 8 9 10 12 14 18 20 23 35 40 60 90 95</td>
</tr>
</tbody>
</table>

Merging a 16-element bitonic sequence through a series of $\log_{16}$ bitonic splits.

The complete network will be based on this component.
We can easily build a sorting network to implement this bitonic merge algorithm.

Such a network is called a \textit{bitonic merging network}.

The network contains \( \log n \) columns. Each column contains \( n/2 \) comparators and performs one step of the bitonic merge.

We denote a bitonic merging network with \( n \) inputs by \( \oplus \text{BM}[n] \).

Replacing the \( \oplus \) comparators by \( \ominus \) comparators results in a decreasing output sequence; such a network is denoted by \( \ominus \text{BM}[n] \).
A bitonic merging network for \( n = 16 \). The input wires are numbered 0, 1, \ldots, \( n - 1 \), and the binary representation of these numbers is shown. Each column of comparators is drawn separately; the entire figure represents a \( \oplus \text{BM}[16] \) bitonic merging network. The network takes a bitonic sequence and outputs it in sorted order.
How do we sort an unsorted sequence using a bitonic merge?

• We must first build a single bitonic sequence from the given sequence.
• A sequence of length 2 is a bitonic sequence.
• A bitonic sequence of length 4 can be built by sorting the first two elements using \( \oplus \text{BM}[2] \) and next two, using \( \ominus \text{BM}[2] \).

❖ This process can be repeated to generate larger bitonic sequences.
A schematic representation of a network that converts an input sequence into a bitonic sequence. In this example, \( \oplus \text{BM}[k] \) and \( \Theta \text{BM}[k] \) denote bitonic merging networks of input size \( k \) that use \( \oplus \) and \( \Theta \) comparators, respectively. The last merging network \( (\oplus \text{BM}[16]) \) sorts the input. In this example, \( n = 16 \).
The comparator network that transforms an input sequence of 16 unordered numbers into a bitonic sequence.
The depth of the network is $\Theta(\log^2 n)$.

Each stage of the network contains $n/2$ comparators. A serial implementation of the network would have complexity $\Theta(n \log^2 n)$. 
Mapping Bitonic Sort to Hypercubes

- Map on a general-purpose parallel computer.
- Consider the case of one item per processor. The question becomes one of how the wires in the bitonic network should be mapped to the hypercube interconnect.
- Note from our earlier examples that the compare-exchange operation is performed between two wires only if their labels differ in exactly one bit!
- A direct mapping of wires to processors; all communication is nearest neighbor!
Communication during the last stage of bitonic sort. Each wire is mapped to a hypercube process; each connection represents a compare-exchange between processes.
Mapping Bitonic Sort to Hypercubes

Communication characteristics of bitonic sort on a hypercube. During each stage of the algorithm, processes communicate along the dimensions shown.
Mapping Bitonic Sort to Hypercubes

Parallel formulation of bitonic sort on a hypercube with $n = 2^d$ processes.

```
1. procedure BITONIC_SORT(label, d)
2. begin
3.     for i := 0 to d - 1 do
4.         for j := i downto 0 do
5.             if $(i + 1)^{st}$ bit of label $\neq$ $j^{th}$ bit of label then
6.                 comp_exchange_max(j);
7.             else
8.                 comp_exchange_min(j);
9.     end BITONIC_SORT
```
During each step of the algorithm, every process performs a compare-exchange operation (single nearest neighbor communication of one word).

Since each step takes $\Theta(1)$ time, the parallel time is

$$T_p = \Theta(\log^2 n)$$

This algorithm is cost optimal w.r.t. its serial counterpart, but not w.r.t. the best sorting algorithm ($\Theta(n \log n)$).
Mapping Bitonic Sort to Meshes

- The connectivity of a mesh is lower than that of a hypercube, so we must expect some overhead in this mapping.

- Consider the row-major shuffled mapping of wires to processors.
Different ways of mapping the input wires of the bitonic sorting network to a mesh of processes: (a) row-major mapping, (b) row-major snakelike mapping, and (c) row-major shuffled mapping.
The last stage of the bitonic sort algorithm for $n = 16$ on a mesh, using the *row-major shuffled mapping*. During each step, process pairs compare-exchange their elements. Arrows indicate the pairs of processes that perform compare-exchange operations.
In the row-major shuffled mapping, wires that differ at the \( i^{th} \) least-significant bit are mapped onto mesh processes that are \( 2^{\lfloor (i-1)/2 \rfloor} \) communication links away.

The total amount of communication performed by each process is:

\[
\sum_{i=1}^{\log n} \sum_{j=1}^{i} 2^{\lfloor (j-1)/2 \rfloor} \approx 7\sqrt{n}, \text{ or } \Theta(\sqrt{n})
\]

The total computation performed by each process is \( \Theta(\log^2 n) \).

The parallel runtime is: \( T_P = \Theta(\log^2 n) + \Theta(\sqrt{n}) \).

This is optimal for the mesh, but not cost optimal.
Block of Elements Per Processor

- Each process is assigned a block of $n/p$ elements.
- The first step is a local sort of the local block.
- Each subsequent compare-exchange operation is replaced by a compare-split operation.
- We can effectively view the bitonic network as having $(1 + \log p)(\log p)/2$ steps $\Rightarrow \Theta(\log^2 p)$.
Initially the processes sort their $n/p$ elements (using merge sort) in time $\Theta((n/p)\log(n/p))$ and then perform $\Theta(\log^2 p)$ compare-split steps.

The parallel run time of this formulation is

$$T_P = \Theta \left( \frac{n}{p} \log \frac{n}{p} \right) + \Theta \left( \frac{n}{p} \log^2 p \right) + \Theta \left( \frac{n}{p} \log^2 p \right).$$
The parallel runtime in this case is given by:

\[
T_P = \Theta \left( \frac{n}{p} \log \frac{n}{p} \right) + \Theta \left( \frac{n}{p} \log^2 p \right) + \Theta \left( \frac{n}{\sqrt{p}} \right)
\]
Performance of Parallel Bitonic Sort

The performance of parallel formulations of bitonic sort for $n$ elements on $p$ processes.

<table>
<thead>
<tr>
<th>Architecture</th>
<th>Maximum Number of Processes for $E = \Theta(1)$</th>
<th>Corresponding Parallel Run Time</th>
<th>Isoefficiency Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypercube</td>
<td>$\Theta(2^{\sqrt{\log n}})$</td>
<td>$\Theta(n/(2^{\sqrt{\log n}}) \log n)$</td>
<td>$\Theta(p^{\log p} \log^2 p)$</td>
</tr>
<tr>
<td>Mesh</td>
<td>$\Theta(\log^2 n)$</td>
<td>$\Theta(n/\log n)$</td>
<td>$\Theta(2^{\sqrt{p}} \sqrt{p})$</td>
</tr>
<tr>
<td>Ring</td>
<td>$\Theta(\log n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(2^p p)$</td>
</tr>
</tbody>
</table>