The Road from Leibniz to Turing, Part I

DA2205, DD3001

October 4, 2012

Leibniz's Dream

Source: Lecture material is based on *The Universal Computer* by Martin Davis

Basic bio Gottfried Leibniz

- **Born**: Leipzig, Germany, July 1646.
- **Died**: Hanover, Germany, Nov 1716.
- **Father**: Professor of Moral Philosophy, University of Leipzig.

  Gottfried had unrestricted access to his father's extensive library.

Gottfried Leibniz: Education and Employment

- **Education**:
  - ⋆ BA in Philosophy, University of Leipzig, 1662
  - ⋆ Master's in Philosophy, Univ. of Leipzig, 1664
  - ⋆ Law degree, University of Leipzig, 1665
  - ⋆ Doctorate in Law, University of Altdorf, 1666

- **Employment**:
  Wealthy noble patrons
  - ⋆ Baron von Boyneburg, 1666 – 1674
    - Diplomatic missions for Elector of Mainz
      $\rightarrow$ Got to spend time in Paris.
  - ⋆ Dukes of Hanover, 1675 – 1716
    - Political adviser, historian, librarian

Gottfried Leibniz: Major Research Achievements

- Prominent figure in the history of mathematics and the history of philosophy.
- Infinitesimal calculus – probably independently of Newton!

  We still use his notation today.
- Towards the development of computation
  - ⋆ Invented mechanical calculator capable of

    $$+, -, \times, \div$$
  - ⋆ Contributions to Formal Logic – unpublished in lifetime
  - ⋆ Believed human reasoning could be reduced to calculations
  - ⋆ Envisaged a *calculus ratiocinator* – resembling symbolic logic – to make such calculations feasible.
  - ⋆ Studied binary notation

Gottfried Leibniz: Fame & fortune during lifetime

- Much.
- Was a major courtier to a powerful German royal dynasty $\rightarrow$ good lifestyle but forced to perform time sapping duties unrelated to his interests – genealogy.
- Corresponded with the major thinkers of his day.
- Made a member of the Royal Society of London in 1673.
- However, reputation towards end of life was in decline.
- Especially tarnished by the controversy with Newton over the discovery of calculus.
- Posthumously though his reputation was restored.

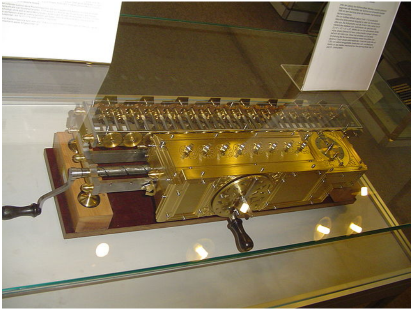
Leibniz's Dream

- As a teenager was introduced to the work of Aristotle.
- This inspired a "*wonderful idea*":
  - Seek an alphabet whose elements represent concepts
  - This alphabet would form a language
  - In this language by symbolic reasoning determine
    - which sentences in the language were true **and**
    - what logical relationships existed among them.
- Leibniz held onto this vision throughout his lifetime...
- ... and made some progress towards it.

Time in Paris

- From 1672-1676 Leibniz was in Paris on a diplomatic mission.
- During this time was exposed to the modern Mathematics of the day which had been fueled by the
  - ⋆ systemization of the techniques for dealing with algebraic expressions
  - ⋆ realization geometry could be expressed as algebra.
- Made contact with the great thinkers of the time.
- Own research:
  - ⋆ Leibniz series for π
  - ⋆ many of the concepts and ideas needed for his derivation of calculus.
- Convinced himself that it is crucial to have an appropriate symbolism when representing and solving problems.

The Stepped Reckoner



- **1671**: Leibniz began work on the "*Stepped Reckoner*" a machine that could $+, -, \times, \div$.
- Prototypes made in Hanover by a craftsman working under Leibniz's supervision.
- Not an unambiguous success as did not fully mechanize the operation of carrying. But its "Leibniz wheel" was a success.

Back to Leibniz's idea

Leibniz saw three strands to his problem:

- Create a compendium of all human knowledge - Crazy then! Crazy now?
- Identify key underlying notions in this compendium and provide them with appropriate symbols
- Rules of deduction encoded as manipulation of these symbols – Leibniz's **calculus ratiocinator**, the algebra of logic.

Leibniz's attempt at symbolic logic

DEFINITION 3. A is in L, or L contains A, is the same as to say that L can be made to coincide with a plurality of terms taken together of which A is one. $B \oplus N = L$ signifies that B is in L and that B and N together compose or constitute L. The same thing holds for a larger number of terms.

AXIOM 1. $B \oplus N = N \oplus B$.

POSTULATE. Any plurality of terms, as A and B, can be added to compose a single term $A \oplus B$.

AXIOM 2. $A \oplus A = A$.

PROPOSITION 5. *If A is in B and $A = C$, then C is in B.*

For in the proposition A *is in* B the substitution of A for B gives C *is in* B.

PROPOSITION 6. *If C is in B and $A = B$, then C is in A.*

For in the proposition C *is in* B the substitution of A for B gives C *is in* A.

PROPOSITION 7. *A is in A.*

For A is in $A \oplus A$ (by Definition 3). Therefore (by Proposition 6) A is in A.

. . . .

PROPOSITION 20 *If A is in M and B is in N, then $A \oplus B$ is in $M \oplus N$.*

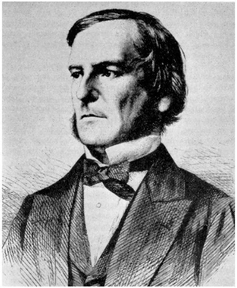
SAMPLE FROM ONE OF LEIBNIZ'S LOGICAL CALCULI

Boole turns logic into algebra

Basic bio George Boole

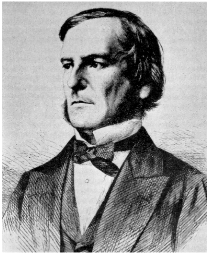
- **Born**: Lincoln, England, Nov 1815.
- **Died**: Cork, Ireland, Dec 1864.
- **Father**: Cobbler and an inept businessman

From the age of 16 George was responsible for providing financially for the family.



George Boole: Education and Employment

- **Education**:
  - ⋆ Elementary school education
  - ⋆ Self-taught with some guidance from the *Lincoln Mechanics' Institution*

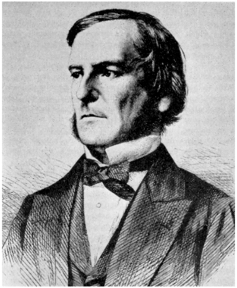


- **Employment**:
  - ⋆ School teacher, $\sim$ Lincoln, 1832–1835
  - ⋆ Ran and taught in schools he founded, Lincoln, 1835–1849
  - ⋆ Professor, University College Cork, Ireland, 1849–1864

Note: Before his professorship he become an active mathematician while running his school !

George Boole: Major Research Achievements

- Boolean Logic - the basis of calculations in the modern digital computer !
- In the *The Laws of Thought* demonstrated that logical deduction could be seen as a branch of mathematics (algebra).
- Also made contributions to differential equations.



George Boole: Fame & Fortune

- Some.
- He received a medal from the *Royal Society* for a publication on linear differential equations.
- Made it to Professor of Mathematics though in a provincial backwater !
- May have progressed further and earlier if he had had a more conventional and privileged background.

Boole was aware that the power of algebra is derived from

- the fact that it has the symbols representing both quantities and operations **and**

- these obey a small number of basic rules or laws.

Classical logic and introduction of symbols

- Classical logic involves sentences such as
  1. All plants are alive.
  2. No hippopotamus is intelligent.
  3. Some people speak English.
- Boole realized that if we reason about the words alive, hippopotamus, or people what is significant for each is the class of all individuals described by the word.
- Boole saw how this reasoning could be expressed in terms of an algebra of such classes.
- Boole represented classes by letters. In his own words...

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Introduction of symbols for classes

*"... If an adjective, as '**good**', is employed as a term of description, let us represent by a letter, as y, all things to which the description '**good**' is applicable, i.e. '**all good things**', or the class '**good things**'. Let it further be agreed, that by the combination xy shall be represented that class of things to which the names of descriptions represented by x and y are simultaneously applicable. Thus, if x alone stands for '**white things**' and y for '**sheep**', let xy stand for '**white sheep**' and in like manner, if z stands for '**horned things**', ...let zxy represent '**horned white sheep**'.*

– *George Boole*

Introduction of symbols for classes

- Following Boole's example and notation, what is yy?

- yy is the class of sheep that are also ...sheep. Therefore

$$yy = y$$

- Most of Boole's system of logic is based on the fact

  when x stands for a class, the equation $xx = x$ is always true

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  when x stands for a class, the equation $xx = x$ is always true

- In ordinary algebra, where x stands for a number, when is the equation $xx = x$ true?

- **Answer**: The equation is true when either $x = 0$ or $x = 1$.

- Boole concluded

  algebra of logic $\equiv$ ordinary algebra restricted to the numbers $\{0, 1\}$

Boole's algebra of logic

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To have an **algebra for logic** similar to **ordinary logic**, need

1. Definition for classes of the binary operators
   - ⋆ Addition

     $$x + y \equiv \text{class consisting of both the elements of } x \text{ and } y$$

   - ⋆ Subtraction

     $$x - y \equiv \text{class consisting of the elements in } x \text{ not in } y$$

2. Identity elements for each binary operator
   - ⋆ **Multiplication**: Need 1 s.t. $1x = x$. 1 is then the class

     "*containing every object under consideration*" $\equiv 1$

   - ⋆ **Addition**: Need a 0 s.t. $x + 0 = x$. 0 is then the class

     "*containing nothing*" $\equiv 0$

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   - ⋆ **Addition**: Need a 0 s.t. $x + 0 = x$. 0 is then the class

     "*containing nothing*" $\equiv 0$

- In this logic what is
  - ⋆ $0x$?
  - ⋆ $1 - x$?
  - ⋆ $x(1 - x)$?
- Note the final expression can be derived from

$$xx = x \implies x - xx = 0 \implies x(1 - x) = 0$$

- In this logic
  - ⋆ $0x = 0$
  - ⋆ $1 - x =$ class "*containing every object not in* x"
  - ⋆ $x(1 - x) = 0$ i.e. nothing can belong **and** not belong to a class

- Note the final expression can be derived from

$$xx = x \implies x - xx = 0 \implies x(1 - x) = 0$$

- Aristotle's logic focused on inferences of a special type called **syllogisms**.
- Inference is from a pair of premises to a conclusion.
- The premises and conclusions must be representable by a sentence of this form

Sentence type
All X are Y
No X are Y
Some X are Y
Some X are not Y

- A **valid** syllogism is

$$(\text{All } X \text{ are } Y) \text{ and } (\text{All } Y \text{ are } Z) \implies (\text{All } X \text{ are } Z)$$

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Boole's algebra and Aristotle's logic

- A **valid** syllogism is

$$(\text{All } X \text{ are } Y) \text{ and } (\text{All } Y \text{ are } Z) \implies (\text{All } X \text{ are } Z)$$

- Translation of premises into Boole's algebra

$$X = XY \quad \text{and} \quad Y = YZ$$

  then the conclusion is

$$X = XY = X(YZ) = (XY)Z = XZ$$

- Translation of conclusion back to words is (All X are Z)

Boole's algebra and Aristotle's logic

- Of course not all syllogism are valid.

- An **invalid** syllogism is

$$(\text{All } X \text{ are } Y) \text{ and } (\text{All } X \text{ are } Z) \implies (\text{All } Y \text{ are } Z)$$

- This invalid syllogism cannot be derived in Boole's logic.

- Therefore Boole's logic includes Aristotle's logic, but is capable of reasoning far beyond it.

Boole and Leibniz's Dream

- ✓ It is easy to use Boole's algebra as a system of rules for calculating $\implies$ provided the calculus ratiocinator.

- ✓ Boole showed logical deduction could be developed as a branch of mathematics.

- ✓ Boole's system of logic included Aristotle's logic and went beyond it.

- ✗ Boole's logic was still fairly crude. Cannot represent statements of the type:

  All failing students are either stupid or lazy

- ✗ Boole's logic did not encompass his logic as a fully-fledged deductive system in which all the rules are deduced from a small set of axioms.

Frege: From breakthrough to despair

Basic bio Gottlob Frege

- **Born**: Wismar, Germany, Nov 1848.
- **Died**: Bad Kleinen (near Wismar), Germany July 1925.
- **Father**: Co-founded and was headmaster of a girls' high school
- **Mother**: Ran the school after Frege's father's death.

Gottlob Frege: Education and Employment

- **Education**:
  - ⋆ University of Jena, 1869 – 1870
  - ⋆ University of Göttingen, 1871 – 1873
  - ⋆ Awarded PhD in Mathematics in 1873

- **Employment**:
  - ⋆ Unpaid lecturer, University of Jena, 1874 – 1879
  - ⋆ Associate Professor, University of Jena, 1879 – 1918

Gottlob Frege: Fame & Fortune

- Not much !
- His achievements were mainly unrecognized in his lifetime.
- Didn't make it to Professor.
- Left broken by his work and embittered at the time of his death.

Gottlob Frege: Major Research Achievements

- Modern logic - axiomatic predicate logic !
- Introduced and discovered how to manipulate the quantifiers ($\forall, \exists$), truth functions ($\neg, \wedge, \vee$ and $\implies$), variables and predicates.
- Developed artificial language with precise rules of grammar $\implies$ logical inferences as purely mechanical operations.
- This logic system encompassed all of the reasoning used by mathematics.
- Frege's logic was an enormous advance over Boole's

Introducing his logical system

BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

FORMELSPRACHE

DES REINEN DENKENS.

VON

Dr. GOTTLOB FREGE,

PRIVATDOCENTEN DER MATHEMATIK AN DER UNIVERSITÄT JENA.

HALLE a/S.

VERLAG VON LOUIS NEBERT.

1879.

In 1879 Frege published

Begriffsschrift - Concept Notation, the Formal Language of the Pure Thought like that of Arithmetics.

outlining his logic system.

Statements in Frege's Logic – Universal quantifier

The sentence

All horses are mammals.

in can be expressed in "Frege speak"

If x is a horse, **then** x is a mammal.

$$\Downarrow$$

$(\forall x)$(**if** x is a horse, **then** x is a mammal).

$$\Downarrow$$

$(\forall x)(x$ is a horse $\supset x$ is a mammal).

$$\Downarrow$$

$(\forall x)(\text{horse}(x) \supset \text{mammal}(x))$.

- These steps of abstract are possible because
  - ⋆ The original statement is true for all x - symbol $\forall$ denotes "*for all*".
  - ⋆ Logical relation **if** ..., **then** ... is symbolized by $\supset$.

Statements in Frege's Logic – Existential quantifier

The sentence

Some horses are pure-bred

in can be expressed in "Frege speak"

x is a horse **and** x is pure-breed.

$$\Downarrow$$

$(\exists x)(x$ is a horse **and** x is pure-bred).

$$\Downarrow$$

$(\exists x)(x \text{ is a horse} \wedge x \text{ is pure-bred})$.

$$\Downarrow$$

$(\exists x)(\text{horse}(x) \wedge \text{pure-bred}(x))$.

- These steps of abstract are possible because
  - ⋆ Original statement is only true for **some** x - $\exists$ denotes "*there exists*".
  - ⋆ Relation **and** is symbolized by $\wedge$.

Symbol	**Interpretation**
$\neg$	not . . .
$\vee$	. . . or . . .
$\wedge$	. . . and . . .
$\supset$	if . . ., then . . .
$\forall$	every
$\exists$	some

- Boole's logic could not express the statement

  All failing students are either stupid or lazy

- How do we express this assertion in Frege's system?

$$(\forall x)(\text{Failing}(x) \supset (\text{Stupid}(x) \vee \text{Lazy}(x)))$$

Some of the symbols of the logic

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$\neg$	not ...
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Frege's rules of inference

Most fundamental rule of inference is as follows

- $\triangle$ is a sentence
- $\Diamond$ is a sentence
- If both

$$\triangle \textbf{ and } (\triangle \supset \Diamond)$$

  are true then $\Diamond$ is true.

Frege's Ambitious Goal

- **Question**:

  Why did Frege develop this logic system?

- **Answer**: Frege believed mathematics is nothing but logic.

  Wanted to show

  - ⋆ given his logic and the concept of set, then all of mathematics follows
  - ⋆ every concept in mathematics can be explicitly defined in terms of logic **and**
  - ⋆ every statement in mathematics can be translated by a well-formed formula of logic.
  - ⋆ all of the basic principles of mathematics could be derived from the fundamental laws of logic.

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Mathematics $\subset$ Logic

- Given the natural numbers can derive much of mathematics

> *"God made natural numbers, all the rest are made by Man.*
>
> *– Leopold Kroneker (1823 - 1891)*

- Therefore, Frege was going to achieve his goal by providing a purely logical theory of the natural numbers - $\{1, 2, 3, \ldots\}$
- He outlined his project in the book *The Foundations of Arithmetic*, 1884.
- He developed his project in detail in the two-volume set *The Fundamental Laws of Arithmetic*. Vol I (1893), Vol II (1903).

Introduction of sets

- To fulfil his goal Frege introduced the concept of a set.
- If b is an element of a set a then write

$$b \in a$$

- Added two principles to basic logic:
  - ⋆ Sets with the same members are the same set:

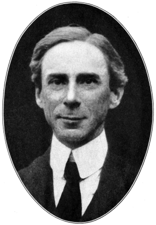
$$x = y \iff \forall z(z \in x \iff z \in y)$$

  - ⋆ Given any property F there is a set consisting of all those things that had F

$$(\exists y)(\forall x)(x \in y \iff F(x))$$

Disaster - A letter in from Bertrand Russell

- In 1902 as his book was going to press Frege received a letter from the young British philosopher Bertrand Russell.



- Russell brought Frege's attention to the axiom of set existence

$$(\exists\, y)(\forall\, x)(x \in y \iff F(x))$$

and asked what happens when $F(x)$ represents the property

"*x is not a member of itself*"

, that is $F(x) \iff \neg(x \in x)$ and x is a set. What happens?

Russell's paradox

- According to Frege there should be a set consisting of all and only those sets that don't belong to themselves:

$$(\exists\, y)(\forall\, x)(x \in y \iff \neg(x \in x))$$

- But if this is true for all x that

$$x \in y \iff \neg(x \in x)$$

then it's true in particular for y:

$$y \in y \iff \neg(y \in y)$$

A contradiction !

- Thus Frege's **Basic principle of logic** was not true.
- **A decade of his work was invalidated.**

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A contradiction !

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Frege – a broken man

> *"There is nothing worse can happen to a scientist than to have the foundation collapse just as the work is finished. I have been placed in this position by a letter from Mr. Bertrand Russell."*
>
> – *Gottlob Frege, Appendix of The Fundamental Laws of Arithmetic, 2nd volume*

- Frege never really recovered from this blow to his work.

- ✓ *Begriffsschrift* can be seen as embodying the universal language of logic envisioned by Leibniz.

- ✓ *Begriffsschrift* encapsulated the logic used in ordinary maths $\implies$ mathematical activity could be investigated by mathematical methods.

- ✗ Frege's logic not efficient for calculations.

- ✗ Frege's logic provides no procedures for determining whether some desired conclusion can be deduced from a set of premises.

Cantor: Detour through infinity

Why this detour?

- Cantor's work marks

  the beginning of the death of certainty in mathematics and
  the birth of computer science !

- Statement based on the fact

Cantor's work raised troubling paradoxes

$\downarrow$

Hilbert & Gödel worked to resolved them

$\downarrow$

Turing was inspired in turn by this work

$\downarrow$

John von Neumann borrowed from Turing to design the EDVAC.

He realized the computing machine is just a logic machine.

Basic bio Georg Cantor

- **Born**: St. Petersburg, Russia, 1845.

  Family moved to Germany when Georg was 11.

- **Died**: Halle, Germany, January 1918.

- **Father**: Successful businessman.

  Inherited sufficient money on father's death to pursue an academic career.

Georg Cantor: Education and Employment

- **Education**:
  - ⋆ University of Zürich, 1862
  - ⋆ University of Berlin, 1863–1867
  - ⋆ University of Göttingen, Summer 1866
  - ⋆ Awarded PhD in Mathematics in 1867
- **Employment**:
  - ⋆ Teacher in a girl's school, Berlin, 1868
  - ⋆ University of Halle, Germany
    - Privatdozent, 1869–1872
    - Extraordinary Professor, 1872–1879
    - Professor, 1879–1913

Georg Cantor: Major Research Achievements

- Inventor of set theory
- Established the importance of one-to-one correspondence between the members of two sets
- Created a profound and coherent mathematical theory of the infinite.

Georg Cantor: Fame & fortune during lifetime

- Some, but had to endure much criticism !
- Full Professor by the age of 34.
- Work not accepted by much of the establishment:
  - ⋆ "grave disease" infecting mathematics, Poincaré
  - ⋆ "scientific charlatan", "corrupter of youth", Kronecker
  - ⋆ "utter nonsense", "laughable", Wittgenstein
- Thus blocked from Professorships at prestigious universities.
- Did have supporters: Dedekind, Weierstrass and Mittag-Leffler.

  Work was "... about one hundred years too soon.", Mittag-Leffler
- Received prestigious accolades later on though.
  - ⋆ 1904 - *Sylvester Medal* from the Royal Society
  - ⋆ 1911 - Honorary Doctorate from St. Andrews University, Scotland

Deciding if two sets have the same size

- Two sets have the same number of members (cardinality) if the members in each set can be matched up in a 1-1 fashion.

- Sets $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and $\{a, b, c, d\}$ have same cardinality as:

$$\begin{array}{cccc} \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a & b & c & d \end{array}$$

- Sets $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and $\{a, b, c, d, e\}$ do not as:

$$\begin{array}{ccccc} \clubsuit & \diamondsuit & \heartsuit & \spadesuit & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a & b & c & d & e \end{array}$$

- Cantor applied the idea of 1-1 matching with infinite sets.

Infinite sets of the same cardinality

- Consider these two sets
  - ⋆ set of all natural numbers, $1, 2, 3, 4, \ldots$ and
  - ⋆ set of all even natural numbers $2, 4, 6, \ldots$

  Do these two sets have the same cardinality?

- **Yes**

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & \cdots \end{array}$$

- Cantor investigated which other infinite sets could be matched in 1-1 correspondence...

- Is the set of rational numbers larger then the set of natural numbers ?
- **No**
- Can list all the possible fractions as follows

$$\left|\frac{1}{1}\right| \frac{1}{2}\frac{2}{1}\left| \frac{1}{3}\frac{2}{2}\frac{3}{1}\right| \frac{1}{4}\frac{2}{3}\frac{3}{2}\frac{4}{1}\left| \frac{1}{5}\frac{2}{4}\frac{3}{3}\frac{4}{2}\frac{5}{1}\right| \cdots$$

Grouped so numerator + denominator equals 2, 3, 4, 5,...

- Matching up is now trivial

$$\begin{array}{cccccccc} \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & \frac{1}{3} & \frac{2}{2} & \frac{3}{1} & \frac{1}{4} & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array}$$

- Is the set of rational numbers larger then the set of natural numbers ?
- **No**
- Can list all the possible fractions as follows

$$\left|\frac{1}{1}\right| \ \frac{1}{2}\frac{2}{1}\left| \ \frac{1}{3}\frac{2}{2}\frac{3}{1}\right| \ \frac{1}{4}\frac{2}{3}\frac{3}{2}\frac{4}{1}\left| \ \frac{1}{5}\frac{2}{4}\frac{3}{3}\frac{4}{2}\frac{5}{1}\right| \cdots$$

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$$\begin{array}{cccccccc} \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & \frac{1}{3} & \frac{2}{2} & \frac{3}{1} & \frac{1}{4} & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{array}$$

Infinite sets can come in different sizes

- Is the set of real numbers larger then the set of natural numbers ?

- **Yes**

- Note though the set of algebraic numbers has the same size as the natural numbers.

- Cantor let
  - ⋆ $\aleph_0$ represent the cardinality of the set of natural numbers
  - ⋆ $\mathcal{C}$ the cardinality of the set of real numbers.

Ranking of members of a set

- Members of $\{\clubsuit, \diamondsuit, \heartsuit\}$ can be ranked in 6 different ways:

1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\diamondsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\heartsuit$	$\clubsuit$	$\heartsuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$

  For each ranking, easy to label the rank of a set member.

- But what about infinite sets? Consider the natural numbers.

- Can obviously list these in any order we like.

- Say all even numbers are listed and then all odd numbers

$$2, 4, 6, \ldots, 1, 3, 5, \ldots$$

  What is the rank of number '2' ?

Ranking of members of a set

- Members of $\{\clubsuit, \diamondsuit, \heartsuit\}$ can be ranked in 6 different ways:

1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}	1^{st}	2^{nd}	3^{rd}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\diamondsuit$	$\clubsuit$	$\heartsuit$	$\diamondsuit$	$\heartsuit$	$\clubsuit$	$\heartsuit$	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$

  For each ranking, easy to label the rank of a set member.

- But what about infinite sets? Consider the natural numbers.
- Can obviously list these in any order we like.
- Say all even numbers are listed and then all odd numbers

$$2, 4, 6, \ldots, 1, 3, 5, \ldots$$

  What is the rank of number '1' ?

Ranking of members of an infinite set

- Cantor introduced the first *transfinite ordinal number* ω.

$$\begin{array}{ccccccccc} 1^{\text{st}} & 2^{\text{nd}} & 3^{\text{rd}} & \cdots & (w+1)^{\text{th}} & (w+2)^{\text{th}} & (w+3)^{\text{th}} & \cdots \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \\ 2 & 4 & 6 & \cdots & 1 & 3 & 5 & \cdots \end{array}$$

- Natural numbers can be ranked in many different ways using larger and larger transfinite ordinal numbers.

- Set $\{$all ordinal numbers needed to define all rankings of $\mathbb{N}\}$ said to have cardinality $\aleph_1$.

- Cantor proved $\aleph_1 > \aleph_0$ and there is no cardinal number $\aleph$ s.t. $\aleph_0 < \aleph < \aleph_1$.

- But it doesn't stop here....

An infinite number of infinite cardinalities

- What about ranking the members of sets of cardinality $\aleph_1$?
- For these rankings need to introduce the transfinite ordinal number ω_1
- Set $\{$all ordinal numbers needed to define all rankings of sets with cardinality $\aleph_1\}$ has cardinality $\aleph_2$.
- There is no end to this process. Can define $\aleph_3, \aleph_4, \ldots, \aleph_\omega, \ldots$

- Cantor used versions of the diagonal method to show
  - ⋆ The cardinality of the set of real numbers is larger than $\aleph_0$
  - ⋆ the cardinality of a set S is less than the cardinality of the power set of S.
- The players in this story also used the diagonal method
  - ⋆ Russell when considering the *set of all sets*.
  - ⋆ Gödel proving his first incompleteness theorem
  - ⋆ Turing in analyzing the *Entscheidungsproblem*.

The diagonal method overview

$$
\begin{array}{lccccccccccccc}
E_0 & = & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \dots \\
E_1 & = & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \mathsf{w} & \dots \\
E_2 & = & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \dots \\
E_3 & = & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \dots \\
E_4 & = & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \dots \\
E_5 & = & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \dots \\
E_6 & = & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \dots \\
E_7 & = & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \dots \\
E_8 & = & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \dots \\
E_9 & = & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \dots \\
E_{10} & = & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \dots \\
E_{11} & = & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
E_u & \neq & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{w} & \mathsf{m} & \mathsf{w} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{m} & \mathsf{w} &
\end{array}
$$

- Cantor's diagonal method for the existence of uncountable sets.
- Bottom sequence cannot occur anywhere in the list of sequences above.
- Therefore cannot list all the infinite sequences of the above form.

- Valid reasoning with Cantor's transfinite can lead to paradoxes.
  - ⋆ **1895**: Cantor considered

    What is the cardinality of the set of all cardinal numbers ?

  - ⋆ **1897**: Burali-Forti published a similar paradox when considering the set of all transfinite ordinal numbers.

- Bertrand Russell while considering Cantor's work asked

  Can there be a set of all sets?

  $\longrightarrow$ led to paradox of sets who are members of themselves

  $\longrightarrow$ letter to Frege.

- Mathematics in crisis.
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Paradoxes !

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